7a)

Uporabom tenzora elastičnosti (ili matrica konstanti)

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

k izraza za

dilataciju volumena $\Delta = u_{k,k}$ $\quad \Theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$\delta_{ij}\delta_{jk} = \delta_{ik}$

tenzor deformacije

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

$$\tau_{ij} = c_{ijkl}\, \varepsilon_{kl}$$

tenzor napetosti

Pokažite da vrijedi:

$$\boxed{\tau_{ij} = \lambda \delta_{ij} \Delta + 2\mu \varepsilon_{ij}}$$

HOOKOV ZAKON

$$\tau_{ij} = c_{ijkl}\, \varepsilon_{kl} =$$

$$= \left[\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] \cdot \frac{1}{2}\left[u_{k,l} + u_{l,k}\right] =$$

$$= \frac{\lambda}{2}\delta_{ij}\Big[\overset{(*)\,= u_{k,k}}{\delta_{kl} u_{k,l}} + \overset{= u_{l,l}}{\delta_{kl} u_{l,k}}\Big] + \frac{1}{2}\mu\Big[\underbrace{\delta_{ik}\delta_{jl} u_{k,l}}_{\overset{(**)}{u_{i,j}}} + \underbrace{\delta_{ik}\delta_{jl} u_{l,k}}_{u_{j,i}} +$$

$$+ \underbrace{\delta_{il}\delta_{jk} u_{k,l}}_{u_{j,i}} + \underbrace{\delta_{il}\delta_{jk} u_{l,k}}_{u_{i,j}}\Big] =$$

$(*)$ $\delta_{kl} u_{k,l} \Rightarrow l = k \Rightarrow u_{k,k}$

$(**)$ $\delta_{ik}\delta_{jl} u_{kl} = \delta_{ik}\left[\delta_{1l}\frac{\partial u_k}{\partial x_l} + \delta_{2l}\frac{\partial u_k}{\partial x_l} + \delta_{3l}\frac{\partial u_k}{\partial x_l}\right] =$

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$$= \left| \text{za } l = j \right| = \delta_{ik} u_{k,j} = \delta_{1k}\frac{\partial u_k}{\partial x_j} + \delta_{2k}\frac{\partial u_k}{\partial x_j} + \delta_{3k}\frac{\partial u_k}{\partial x_j}$$

$$= \left| \text{za } k = i \right| = u_{i,j}$$

$$= \frac{\lambda}{2}\delta_{ij}\, 2\, u_{k,k} + 2\mu \frac{1}{2}\left[u_{i,j} + u_{j,i}\right] = \lambda \delta_{ij}\Delta + 2\mu\varepsilon_{ij}$$

7b)

Pokažite da iz sljedećih izraza

pomak
$$\vec{u} = \nabla \varphi + \vec{\nabla} \times \vec{\Psi} \quad , \quad \vec{\nabla}\vec{\Psi} = 0$$

jednadžba gibanja
$$\ddot{\vec{u}} = \left(\frac{\lambda+2\mu}{\rho}\right)\nabla(\vec{\nabla}\vec{u}) - \left(\frac{\mu}{\rho}\right)\vec{\nabla}\times(\vec{\nabla}\times\vec{u})$$

slijedi izraz
$$\nabla\left\{\frac{\lambda+2\mu}{\rho}\nabla^2\varphi - \ddot{\varphi}\right\} + \vec{\nabla}\times\left\{\frac{\mu}{\rho}\nabla^2\vec{\Psi} - \ddot{\vec{\Psi}}\right\} = 0$$

$$\ddot{\vec{u}} = \nabla\ddot{\varphi} + \vec{\nabla}\times\ddot{\vec{\Psi}}$$

divergencija pomaka
$$\vec{\nabla}\cdot\vec{u} = \vec{\nabla}(\nabla\varphi + \vec{\nabla}\times\vec{\Psi}) = \nabla^2\varphi$$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{\Psi})$

rotacija pomaka
$$\vec{\nabla}\times\vec{u} = \vec{\nabla}\times(\underbrace{\nabla\varphi}_{\vec{\nabla}\times(\nabla\varphi)=0} + \vec{\nabla}\times\vec{\Psi}) = \nabla\underbrace{\vec{\nabla}\vec{\Psi}}_{\vec{\nabla}\vec{\Psi}=0} - \nabla^2\vec{\Psi} = -\nabla^2\vec{\Psi}$$

$$\ddot{\vec{u}} = \nabla\ddot{\varphi} + \vec{\nabla}\times\ddot{\vec{\Psi}} = \frac{\lambda+2\mu}{\rho}\nabla(\nabla^2\varphi) - \frac{\mu}{\rho}\vec{\nabla}\times(-\nabla^2\vec{\Psi})$$

$$\nabla\left(\frac{\lambda+2\mu}{\rho}\nabla^2\varphi - \ddot{\varphi}\right) + \vec{\nabla}\times\left(\frac{\mu}{\rho}\nabla^2\vec{\Psi} - \ddot{\vec{\Psi}}\right) = 0$$

uz pp ρ se ne mijenja

8)

Za valove koji se rasprostiru u proizvoljnom smjeru definiranom vektorom valnog broja $\vec{k}$

a) Pokažite da pomak uzrokovan prolazom P-vala kojeg karakterizira skalarni potencijal $\varphi(\vec{x},t) = e^{i(\omega t - \vec{k}\vec{x})}$ je paralelan sa smjerom rasprostiranja vala.
b) Pokažite da je pomak S-vala karakteriziranog vektorskim potencijalom $\vec{\psi}(\vec{x},t) = Ae^{i(\omega t - \vec{k}\vec{x})}$, $\vec{A} = \left(A_x, A_y, A_z\right)$ okomit na smjer rasprostiranja vala.

a) Za skalarni potencijal

$$\phi(\mathbf{x},\ t) = e^{\,i(\omega t - \mathbf{k}\cdot\mathbf{x})} = e^{\,i(\omega t - k_x x - k_y y - k_z z)},$$

je pomak S-vala:

$$\mathbf{u}(\mathbf{x},\ t) = \nabla\phi(\mathbf{x},t) = \frac{\partial\phi(\mathbf{x},t)}{\partial x}\,\hat{\mathbf{e}}_1 + \frac{\partial\phi(\mathbf{x},t)}{\partial y}\,\hat{\mathbf{e}}_2 + \frac{\partial\phi(\mathbf{x})}{\partial z}\,\hat{\mathbf{e}}_3$$

$$= (-k_x, -k_y, -k_z)e^{\,i(\omega t \pm \mathbf{k}\cdot\mathbf{x})} = -\mathbf{k}e^{\,i(\omega t \pm \mathbf{k}\cdot\mathbf{x})},$$

vektor paralelan s **k**

b) Kako bi pokazali da je pomak S-vala **u** okomit na smjer rasprostiranja vala **k** treba pokazati da je $\mathbf{u}\cdot\mathbf{k} = 0$:

$$\mathbf{u} = \nabla\times\psi = \nabla\times(A_x,\ A_y,\ A_z)e^{\,i(\omega t - k_x x - k_y y - k_z z)} =$$

$$= (-k_y A_z + k_z A_y)\hat{\mathbf{e}}_1 + (-k_z A_x + k_x A_z)\hat{\mathbf{e}}_2 + (-k_x A_y + k_y A_x)\hat{\mathbf{e}}_3,$$

odakle slijedi:

$$\mathbf{u}\cdot\mathbf{k} = -k_x k_y A_z + k_x k_z A_y - k_y k_z A_x + k_y k_x A_z - k_z k_x A_y + k_z k_y A_x = 0.$$