(1) $u = 2\frac{x}{y} + 2\frac{xy^3}{z^2} + 3xz^4$

$\vec{B} = \frac{z}{x}\hat{x} + 2y^3z\hat{y} + 2\frac{y^3z^3}{x^2}\hat{z}$

grad. u , divergencija $\vec{B}$, Laplacian od u , rot grad u

∇u , $\nabla\cdot\vec{B}$, $\nabla^2 u$, $\nabla\times(\nabla u)$

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

---

grad skalarnog poqa = vektorsko poqe

a) $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) =$

$= \left(\frac{2}{y} + \frac{2y^3}{z^2} + 3z^4, -2\frac{x}{y^2} + 6\frac{xy^2}{z^2}, -4\frac{xy^3}{z^3} + 12xz^3\right)$

div vektor. poqa ⇒ skalarno p.

b) $\nabla\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = -\frac{z}{x^2} + 6y^2z + 6\frac{y^3z^2}{x^2}$

c) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial x}(\nabla u)_x + \frac{\partial}{\partial y}(\nabla u)_y + \frac{\partial}{\partial z}(\nabla u)_z$

$= 4\frac{x}{y^3} + 12\frac{xy}{z^2} + 12\frac{xy^3}{z^4} + 36xz^2$

d)

$$\nabla\times(\nabla u) = \det\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \hat{x} & \hat{y} & \hat{z} \end{bmatrix} =$$

$$= \left(\frac{\partial^2}{\partial y\partial z} - \frac{\partial^2}{\partial z\partial y}\right)u\hat{x} + \left(\frac{\partial^2}{\partial z\partial x} - \frac{\partial^2}{\partial x\partial z}\right)u\hat{y} + \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)u\hat{z} =$$

$= 0$

DZ:

$\vec{B}\cdot(\vec{B}\times\nabla u)$

$$= \vec{B}\begin{bmatrix} B_x & B_y & B_z \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \hat{x} & \hat{y} & \hat{z} \end{bmatrix} = \ldots = 0$$

$\mathrm{div}(\mathrm{rot}\,\vec{B}) = 0$

(2)

Na koji način možemo dobiti Helmholtzovu jednadžbu iz valne jednadžbe?

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \rightarrow \text{valna jednadžba}$$

Fourier transform:

$$S(\vec{r},\omega) = \int_{-\infty}^{\infty} \psi(\vec{r},t)\, e^{-i\omega t}\, dt$$

INVERZNI F. transf.

$$\psi(\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega$$

UVRSTIMO U IZRAZ ZA INVERZNI F.T. U VALNU JEDNADŽBU:

$$\cancel{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \nabla^2 \left[S(\vec{r},\omega)\, e^{i\omega t}\, d\omega \right] = \frac{1}{c^2} \cancel{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} \left[S(\vec{r},\omega)\, e^{i\omega t}\, d\omega \right]$$

$$\int_{-\infty}^{\infty} \nabla^2 S(\vec{r},\omega)\, e^{i\omega t}\, d\omega = \frac{1}{c^2} \int_{-\infty}^{\infty} S(\vec{r},\omega) \frac{\partial^2}{\partial t^2} \left(e^{i\omega t} \right) d\omega$$

$$\int_{-\infty}^{\infty} \nabla^2 S(\vec{r},\omega)\, e^{i\omega t}\, d\omega = -\underbrace{\frac{\omega^2}{c^2}}_{=k_c} \int_{-\infty}^{\infty} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega$$

vektor valnog broja je $\vec{k} = k_c \vec{p}$

$$\int_{-\infty}^{\infty} \nabla^2 S(\vec{r},\omega)\, e^{i\omega t}\, d\omega + k_c^2 \int_{-\infty}^{\infty} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega = 0$$

$$\int_{-\infty}^{\infty} \left[\nabla^2 S(\vec{r},\omega) + k_c^2 S(\vec{r},\omega) \right] e^{i\omega t}\, d\omega = \underbrace{\int_{-\infty} F(\vec{r},\omega)\, e^{i\omega t}\, d\omega}_{\text{spektar f.cije } f(\vec{r},t)} = f(\vec{r},t) = 0$$

Ovdje je $F(\vec{r},\omega)$ spektar f.cije $f(\vec{r},t)$, pa ako je $f(\vec{r},t) = 0$ tada je i $F(\vec{r},\omega) = 0$

(ako nema f.cije nema ni spektra!)

$$\Rightarrow \boxed{\nabla^2 S(\vec{r},\omega) + k_c^2 S(\vec{r},\omega) = 0}$$

HELMHOLTZova jednadžba

1

③ Dokažite da je rješenje Helmholtzove jednadžbe povezano s rješenjem valne jednadžbe sljedećom relacijom:

$$\psi(\vec{r},t) = \frac{1}{\pi} \,\mathrm{Re} \int_0^\infty S(\vec{r},\omega)\, e^{i\omega t}\, d\omega \qquad !$$

Helmholtzova jednadžba:

$$\boxed{\nabla^2 S + k_c^2 S = 0}$$

Rješenja Helmholtzove jednadžbe povezana su s rješenjima valne jednadžbe preko INVERZNOG Four. transforma:

$$\psi(\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega \qquad (1)$$

S obzirom da je $\psi(\vec{r},t)$ realna f.cija tada vrijedi:

$$\psi = \psi^*$$

zvjezdica označava da je riječ o konjugirano kompleksnoj veličini.

Pa slijedi: $S(\vec{r},-\omega) = S^*(\vec{r},\omega)$

DOKAZ

$$S(\vec{r},\omega) = \int_{-\infty}^{\infty} \psi(\vec{r},t)\, e^{-i\omega t}\, dt \quad \Big/^{*}$$

$$\left.\begin{aligned} S^*(\vec{r},\omega) &= \int_{-\infty}^{\infty} \psi(\vec{r},t)\, e^{i\omega t}\, dt \\ S(\vec{r},-\omega) &= \int_{-\infty}^{\infty} \psi(\vec{r},t)\, e^{i\omega t}\, dt \end{aligned}\right\} S^*(\vec{r},\omega) = S(\vec{r},-\omega)$$

Iz (1) slijedi:

$$\psi(\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{0} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega + \frac{1}{2\pi} \int_0^{\infty} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega =$$

$$= \frac{1}{2\pi} \left[\int_0^{\infty} S(\vec{r},-\omega)\, e^{-i\omega t}\, d\omega + \int_0^{\infty} S(\vec{r},\omega)\, e^{i\omega t}\, d\omega \right] = \qquad (2)$$

$$= \frac{1}{2\pi} (I_1 + I_2)$$

ako pišem pojednostavljeno bez $\vec{r}$

$$I_1 = \int s(\omega) e^{i\omega t} d\omega = \int (a+ib)(\cos\omega t + i\sin\omega t) d\omega =$$

$$= \int [(a\cos\omega t - b\sin\omega t) + i\cdot \cancel{(b\cos\omega t + a\sin\omega t)}] d\omega$$

$$I_2 = \int \overset{s^*(\omega)}{s(-\omega)} e^{-i\omega t} d\omega = \int (a-ib)(\cos\omega t - i\sin\omega t) d\omega =$$

$$= \int [(a\cos\omega t - b\sin\omega t) - i\cdot \cancel{(b\cos\omega t + a\sin\omega t)}] d\omega$$

$\Rightarrow$ imaginarni dijelovi od I_1 i I_2 se poništavaju

pa je $\frac{1}{\cancel{2}\pi} \cancel{2} \mathrm{Re} \int_0 s(\omega) e^{i\omega t} d\omega$

$$\boxed{\psi(\vec{r},t) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty s(\vec{r},\omega) e^{i\omega t} d\omega}$$

④ Dokažite da se Navierova jednadžba, uz uporabu Láméova teorema, može pisati u obliku dviju separiranih valnih jednadžbi 4. reda!

$\vec\nabla(\vec\nabla\vec u) \equiv$ grad div $\vec u$

$\vec\nabla\times(\vec\nabla\times\vec u) \equiv$ rot rot $\vec u$

$\vec\nabla\vec u \equiv$ div $\vec u$

$\vec\nabla u \equiv$ grad u

Navierova jdn.

$$\rho\ddot{\vec u} = \vec f + (\lambda+2\mu)\nabla(\nabla\vec u) - \mu\vec\nabla\times(\vec\nabla\times\vec u)$$

gdje je (iz Láméova teorema):

$$\vec f = \vec\nabla\phi + \vec\nabla\times\vec\xi$$
$$\vec\nabla\cdot\vec\xi = 0$$
$$\vec u = \vec\nabla\varphi + \vec\nabla\times\vec\Psi$$
$$\vec\nabla\vec\Psi = 0$$
$$\ddot\varphi = \frac{\phi}{\rho} + \alpha^2\nabla^2\varphi, \quad \alpha^2 = \frac{\lambda+2\mu}{\rho}$$
$$\ddot{\vec\Psi} = \frac{\vec\xi}{\rho} + \beta^2\nabla^2\vec\Psi, \quad \beta^2 = \frac{\mu}{\rho}$$

Helmholtzov teorem kaže da se svako vektorsko polje $\vec u$ može prikazati kao suma bezrotacijske i bezdivergentne komponente

$\vec\nabla\cdot(\vec\nabla\times\vec\Psi) = 0$

$\vec\nabla\times\vec\nabla\varphi = 0$

α, β su brzine rasprostiranja valova φ i $\vec\Psi$

$\vec\nabla\times(\vec\nabla\times\vec\Psi) = \vec\nabla(\vec\nabla\vec\Psi) - \nabla^2\vec\Psi$

$$\rho\vec\nabla\ddot\varphi + \rho(\vec\nabla\times\ddot{\vec\Psi}) = \vec\nabla\phi + \vec\nabla\times\vec\xi + (\lambda+2\mu)\vec\nabla\left[\nabla^2\varphi + \underbrace{\vec\nabla\cdot(\vec\nabla\times\vec\Psi)}_{=0}\right] -$$

$$-\mu\vec\nabla\times\left[\underbrace{\vec\nabla\times(\vec\nabla\varphi)}_{=0} + \underbrace{\vec\nabla\times(\vec\nabla\times\vec\Psi)}_{\underbrace{\vec\nabla(\vec\nabla\vec\Psi)}_{=0} - \nabla^2\vec\Psi}\right]$$

$$\vec\nabla\left[\rho\ddot\varphi - \phi - (\lambda+2\mu)\nabla^2\varphi\right] + \vec\nabla\times\left[\rho\ddot{\vec\Psi} - \vec\xi - \mu\nabla^2\vec\Psi\right] = 0$$

→ parcijalne diferencijalne jednadžbe 3. reda po φ i $\vec\Psi$

Na ovo još moramo primijeniti operatore $\vec\nabla\cdot(\ )$ i $\vec\nabla\times(\ )$ da dobijemo separirane jednadžbe za potencijale i time dobivamo valne jednadžbe 4. reda

Dokažite da se Navierova jednadžba, uz upotrebu Lámeova teorema može pisati u obliku dviju separiranih valnih jednadžbi 4. reda!

Navierova jedn.:

→ ako polje pomaka $\vec{u} = \vec{u}(\vec{x}, t)$ zadovoljava sljedeću jednadžbu:

$$\rho \ddot{\vec{u}} = \vec{f} + (\lambda + 2\mu)\nabla(\nabla \vec{u}) - \mu \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) \qquad (*)$$

i ako su vanjska sila i polje pomaka prikazani u izrazima Helmholtzovog potencijala tada vrijedi Lámeov teorem

f = vanjska sila prikazana pomoću potencijala:

$$\vec{f} = \vec{\nabla}\Phi + \vec{\nabla} \times \vec{\xi}$$

$$\vec{\nabla}\vec{\xi} = 0$$

Helmholtzov teorem kaže da se svako vektorsko polje $\vec{u}$ može prikazati kao suma bezrotacijske i bezdivergentne komponente

polje pomaka

$$\vec{u} = \vec{\nabla}\varphi + \vec{\nabla} \times \vec{\Psi}$$

→ longitudinalna i transverzalna komponenta pomaka

$$\vec{\nabla}\vec{\Psi} = 0$$

valne jednadžbe

$$\ddot{\varphi} = \frac{\Phi}{\rho} + \alpha^2 \nabla^2 \varphi, \quad \alpha^2 = \frac{\lambda + 2\mu}{\rho}$$

$$\ddot{\vec{\Psi}} = \frac{\vec{\xi}}{\rho} + \beta^2 \nabla^2 \vec{\Psi}, \quad \beta^2 = \frac{\mu}{\rho}$$

α, β brzine rasprostiranja valova φ i $\vec{\Psi}$

φ i Ψ su ELASTIČNI POTENCIJALI, ne Helmholtzovi

vektorski identiteti:

$$\vec{\nabla}(\vec{\nabla} \times \vec{\Psi}) = 0$$

$$\vec{\nabla} \times (\vec{\nabla}\varphi) = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\Psi}) = \vec{\nabla}(\vec{\nabla}\vec{\Psi}) - \nabla^2 \vec{\Psi}$$

---

primjenimo ovo na (*) Navierovu j.dbu →

$$\rho\vec\nabla\ddot\varphi + \rho(\vec\nabla\times\ddot{\vec\psi}) = \vec\nabla\phi + \vec\nabla\times\vec\xi +$$

$$+(\lambda+2\mu)\vec\nabla\left[\nabla^2\varphi + \underbrace{\vec\nabla(\vec\nabla\times\vec\psi)}_{=0}\right] -$$

$$-\mu\vec\nabla\times\left[\underbrace{\vec\nabla\times(\vec\nabla\varphi)}_{=0} + \vec\nabla\times(\vec\nabla\times\vec\psi)\right]$$

$$\vec\nabla\times(\vec\nabla\times\vec\psi) = \underbrace{\vec\nabla(\vec\nabla\vec\psi)}_{=0} - \nabla^2\vec\psi$$

i kada to uvrstimo:

$$\vec\nabla\left[\rho\ddot\varphi - \phi - (\lambda+2\mu)\nabla^2\varphi\right] + \vec\nabla\times\left[\rho\ddot{\vec\psi} - \vec\xi - \mu\nabla^2\vec\psi\right] = 0 \qquad (**)$$

to su parcijalne diferencijalne j.džbe 3. reda po φ i $\vec\psi$

Na ovo još moramo primjeniti operatore $\vec\nabla\cdot(\ )$ i $\vec\nabla\times(\ )$ da dobijemo separirane jednadžbe za potencijale ⇒ time dobivamo valne jednadžbe 4. reda!

$(**)/\cdot\vec\nabla \Rightarrow \vec\nabla\times[\ ]=0 \Rightarrow \nabla^2\left[\rho\ddot\varphi - \phi - (\lambda+2\mu)\nabla^2\varphi\right] = 0$

$(**)/\times\vec\nabla \Rightarrow \vec\nabla[\ ]=0 \Rightarrow \vec\nabla\times\left(\vec\nabla\times\left[\rho\ddot{\vec\psi} - \vec\xi - \mu\nabla^2\vec\psi\right]\right) = 0$

SEPARIRALI SMO POREMEĆAJE

$$\vec\nabla\cdot\underbrace{\left[\vec\nabla\cdot(\rho\ddot{\vec\psi} - \vec\xi - \mu\nabla^2\vec\psi)\right]} - \nabla^2(\rho\ddot{\vec\psi} - \vec\xi - \mu\nabla^2\psi) = 0$$

$$\rho\vec\nabla\ddot{\vec\psi} - \underbrace{\vec\nabla\vec\xi}_{=0} - \mu\nabla^2\underbrace{(\vec\nabla\vec\psi)}_{=0} = \rho\vec\nabla\left(\frac{\vec\xi}{\rho} + \beta^2\nabla^2\psi\right) =$$

$$= \underbrace{\vec\nabla\vec\xi}_{=0} + \rho\beta\nabla^2\underbrace{(\nabla\vec\psi)}_{=0} = 0$$

time smo dobili:

⇒ SEPARIRANE VALNE J.DŽBE 4. reda

$$\nabla^2\left(\rho\ddot\varphi - \phi - (\lambda+2\mu)\nabla^2\varphi\right) = 0$$

$$\nabla^2\left(\rho\ddot{\vec\psi} - \vec\xi - \mu\nabla^2\psi\right) = 0$$

TO JE RAZLOG ZAŠTO ELASTIČKE POTENCIJALE NE DEFINIRAMO DA BUDU HELMHOLTZOVI POTENCIJALI.

Lamov teorem nam daje puno bolji rezultat jer tražimo potencijale koji zadovoljavaju valne jednadžbe samo 2. reda !!

6. Dokažite da elastični potencijali φ i $\vec{\psi}$ zadovoljavaju, svaki ponaosob, jednadžbu:

$$\nabla^2\left(\frac{\partial^2 y}{\partial t^2} - c^2\nabla^2 y\right)=0 \qquad y=\varphi, \vec{\psi}$$

gdje je ($\underline{\underline{y}}$ ili $\underline{\underline{\varphi}}$ ili $\underline{\underline{\vec{\psi}}}$, a $\underline{\underline{c}}$ brzina rasprostiranja vala!

$$\nabla^2\left(\frac{\partial^2 y}{\partial t^2} - c^2\nabla^2 y\right)=0$$

Laplace

$$\nabla^2\left(\frac{\partial^2 \varphi}{\partial t^2} - c^2\nabla^2 \varphi\right) = \frac{\partial^2}{\partial t^2}\underbrace{(\nabla^2\varphi)}_{=\Theta} - c^2\nabla^2\underbrace{(\nabla^2\varphi)}_{=\Theta} = \frac{\partial^2\Theta}{\partial t^2} - c^2\nabla^2\Theta = 0$$

(*)

Θ = dilatacija volumena = $\vec{\nabla}\vec{u}$ (dilataciona komponenta pomaka)

$$\nabla^2\left(\frac{\partial^2 \vec{\psi}}{\partial t^2} - c^2\nabla^2 \vec{\psi}\right) = \left| \begin{array}{l} (\ast\ast) \text{ rotaciona komp. pomaka} = \vec{\omega} = \vec{\nabla}\times\vec{u} = \\ = \vec{\nabla}\times(\vec{\nabla}\times\vec{\psi}) = \vec{\nabla}\underbrace{(\vec{\nabla}\vec{\psi})}_{=0} - \nabla^2\vec{\psi} = -\nabla^2\vec{\psi} \end{array} \right| =$$

$$= \frac{\partial^2}{\partial t^2}\underbrace{(\nabla^2\vec{\psi})}_{-\vec{\omega}} - c^2\nabla^2\underbrace{(\nabla^2\vec{\psi})}_{-\vec{\omega}} = -\left[\frac{\partial^2\vec{\omega}}{\partial t^2} - c^2\nabla^2\vec{\omega}\right] = 0$$

Helmholtz:

Svako vektorsko polje se može prikazati kao suma bezrotacijske i bezdivergentne komponente:

$$\vec{u} = \vec{\nabla}\varphi + \vec{\nabla}\times\vec{\psi} \qquad \text{vektorski identiteti:} \quad \vec{\nabla}(\vec{\nabla}\times\vec{\psi}) = 0$$
$$\vec{\nabla}\cdot\vec{\psi} = 0 \qquad \vec{\nabla}\times\vec{\nabla}\varphi = 0$$

(*) $\Theta = \vec{\nabla}\vec{u} = \vec{\nabla}(\vec{\nabla}\varphi + \vec{\nabla}\times\psi) = \vec{\nabla}^2\varphi + \vec{\nabla}\cdot(\vec{\nabla}\times\psi) = \underline{\underline{\vec{\nabla}^2\varphi}}$

(**) $\omega = \vec{\nabla}\times\vec{u} = \vec{\nabla}\times(\vec{\nabla}\varphi + \vec{\nabla}\times\vec{\psi}) = \vec{\nabla}\times(\vec{\nabla}\times\vec{\psi}) = \vec{\nabla}(\vec{\nabla}\vec{\psi}) - \nabla^2\vec{\psi} = -\nabla^2\psi$